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Hence,

$$\sum_{i=1}^{i=k} s_i = n^2\sum_{i=1}^{i=k} i - \frac{kn^2}{2} + \frac{kn}{2} = \frac{n^2k(k+1) - kn^2 + kn}{2} = \frac{nk(nk+1)}{2}$$

438. Proposed by WALTER C. EELLS, U. S. Naval Academy, Annapolis, Maryland.

In Hardy's *Pure Mathematics* (page 14, Nos. 2, 3) occurs the problem: "Show that if m/n is a good approximation to $\sqrt{2}$, then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions, *e. g.*, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, $\cdots$." Find (*a*) other approximations for $\sqrt{2}$ of the same type, *i. e.*,

$$\frac{m'}{n'} = \frac{am + bn}{cm + dn}, \quad (a, b, c, d, m, n, \text{integers}).$$

(*b*) Similar approximations for the square roots of other integers.

Solution by Elijah Swift, University of Vermont.

Assume that $(m/n) < \sqrt{2}$. We wish to find a, b, c, d so that

$$0 < \frac{am + bn}{cm + dn} - \sqrt{2} < \sqrt{2} - \frac{m}{n}. \tag{1}$$

Assume a, b, c, d positive. The second inequality leads, after simple reductions and squaring both sides, to

$$\begin{aligned} c^2m^4 + 2(ac + cd)m^3n + (a^2 + 2ad + d^2 + 2bc - 8c^2)m^2n^2 & \\ + 2(ab + bd - 8dc)mn^3 + (b^2 - 8d^2)n^4 &< 0. \end{aligned} \tag{2}$$

If we had assumed $m/n > \sqrt{2}$, inequality (1) would have been reversed, and hence (2) also. But if the right-hand side of (2) changes sign with $m - \sqrt{2}\,n$, it must have $m - \sqrt{2}\,n$ and hence also $m^2 - 2n^2$ as a factor. This leads to the following equations connecting a, b, c, d.

$$ab + bd + 2ac - 6cd = 0. \tag{3}$$

$$b^2 - 6d^2 + 2a^2 + 4ad + 4bc - 12c^2 = 0. \tag{4}$$

These equations have for a solution $a = d$, $b = 2c$; and this is the only solution in positive integers.

The first inequality of (1) leads, after substitution of a for d, and $2c$ for b, to $(a^2 - 2c^2)(m^2 - 2n^2) > 0$. Hence, $a^2 < 2c^2$. Finally, in (2) putting $a = d$, $b = 2c$, we obtain, after factoring, $[m^2 - 2n^2][c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2] < 0$. Hence, $c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2 > 0$, or $(cm + 2an)^2 > 2c^2n^2$, that is, $m/n > \sqrt{2} - (2a/c)$. If $(2a/c) \geqq 1$, m/n need only be as large as .5 to satisfy this inequality.

If we had assumed $m/n > \sqrt{2}$, we should obviously be led to the same results.

If, then, $m'/n' = (am + 2cn)/(cm + an)$, and $2a \geqq c$, $a^2 < 2c^2$, or in other words $8c^2 > 4a^2 \geqq c^2$, we shall have satisfied the conditions of the problem. Such sets of values are $\begin{cases} c = 1 \\ a = 1 \end{cases}$, $\begin{cases} c = 2 \\ a = 1 \end{cases}$, $\begin{cases} c = 3 \\ a = 2 \text{ or } 4 \end{cases}$, etc. The first set is the one given in the problem. The second gives $m'/n' = (m + 4n)/(2m + n)$ which yields the series 1/1, 5/3, 17/13, 69/47, ...

(*b*) Almost the same work leads to similar results in the case of any surd, $\sqrt{k}$. We find $a = d$, $b = kc$, $2a \geqq c(k - 1)$, $kc^2 > a^2$. These possess, among others, the following solutions for the case where $k = 3$.

$$\begin{cases} c = 1 \\ a = 1 \end{cases}, \begin{cases} c = 2 \\ a = 3 \end{cases}, \begin{cases} c = 3 \\ a = 4 \text{ or } 5 \end{cases}, \text{ etc.}$$

The first set gives $m'/n' = (m + 3n)/(m + n)$, which yields the series 1/1, 4/2 = 2/1, 5/3, 14/8 = 7/4, 19/11, ...

Also solved by NORMAN ANNING.

GEOMETRY.

466. Proposed by HORACE OLSON, Chicago, Illinois.

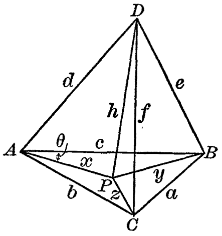
Given the edges of a triangular pyramid, find the radius of the inscribed sphere.

SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Calling the radius of the sphere r, and using the notation in the figure, we have,

$$h^2 = d^2 - x^2 = c^2 - y^2 = f^2 - z^2,$$

$$\cos\theta = \frac{c^2 + x^2 - y^2}{2cx} = \frac{c^2 + d^2 - e^2}{2cx} = \frac{n}{x}. \qquad (x \cos\theta = n.)$$



$$\cos(A - \theta) = \cos A \cos\theta + \sin A \sin\theta = \frac{b^2 + x^2 - z^2}{2bx} = \frac{b^2 + d^2 - f^2}{2bx} = \frac{m}{x}.$$

Hence,

$$\sin\theta = \frac{m - n\cos A}{x \sin A}, \qquad x\sin\theta = \frac{m - n\cos A}{\sin A}.$$

Hence,

$$x^2 = n^2 + \frac{(m - n\cos A)^2}{\sin^2 A}, \quad \text{and} \quad h = \frac{\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{\sin A}.$$

Hence,

$$3 \text{ times contents of pyramid} = \frac{bc}{2}\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2},$$

and therefore,

$$r = \frac{bc\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{2(ABC + ACD + ABD + BCD)},$$